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LETTER TO THE EDITOR

Discussion of "A one-dimensional study of the limit cases of the endochronic theory", *Int. J. Solids Structures*, Vol. 25, No. 8, pp. 863–877 (1989)

1. INTRODUCTION

In a recent paper, entitled "A one-dimensional study of the limit cases of the endochronic theory", Fazio (1989) discussed the one-dimensional version of the endochronic theory in the case where the intrinsic time is the length of the plastic strain path in the plastic strain space. His results have serious negative implications regarding the validity of the theory proposed earlier by Valanis (1980). This letter is to put on record that Fazio's analysis is flawed and his negative conclusions erroneous. As pointed out in a paper by Wu and Komarakulnanakorn (1991) accepted by the *International Journal of Solids and Structures*, *Fazio's entire paper is based on a process of dividing zero by zero. As we point out in this communication, he, in doing so, obtains a result that violates the very inequality that was the basis of his derivation.* The ensuing analysis will demonstrate the point.

It is of historical interest that this issue, with one internal variable, was addressed by Valanis (1978), but the details of the analysis were omitted in a dual paper by Valanis (1980).

2. ANALYSIS

In what follows Arabic numerals will denote Fazio's (1989) equations as they appear in his paper, while Roman numerals will denote equations pertaining to this communication. We begin with Fazio's eqn (17) with the correct sign,

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon + \lambda\{E_\infty\varepsilon - \sigma\}\,\mathrm{d}z \tag{17}$$

if λ is to be a positive material parameter (as it is), though its sign is not relevant to the discussion. For a derivation of this equation see the Appendix. We consider the case in dispute in which:

$$\mathrm{d}z = |\mathrm{d}\varepsilon^p| = |\mathrm{d}\varepsilon - \mathrm{d}\sigma/E_0| \tag{18}$$

where for purposes of demonstration we address the simplest situation where $P = f = 1$. We hasten to add that the form of P, or f, is irrelevant to the argument. Fazio considers three cases:

(a)	$\mathrm{d}\varepsilon > \mathrm{d}\sigma/E_0$	(plastic loading)	(19)
(b)	$\mathrm{d}\varepsilon < \mathrm{d}\sigma/E_0$	(plastic unloading/reverse loading)	(20)
(c)	$\mathrm{d}\varepsilon = \mathrm{d}\sigma/E_0$	(elastic range).	(21)

Case (c) is that of an elastic response when the end point of the stress vector lies within the yield surface. We remind the reader that when the number of internal variables is finite then a yield surface is shown to exist (Valanis, 1980).

We consider Case (a) first and substitute eqn (18) in eqn (17) to obtain:

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon + \lambda(E_\infty\varepsilon - \sigma)(\mathrm{d}\varepsilon - \mathrm{d}\sigma/E_0). \tag{i}$$

This is the equation that led Fazio to erroneous conclusions. We shall first demonstrate that this equation leads to physically satisfactory plastic behavior and then proceed to show the fallacy that underlies Fazio's argument.

When terms in eqn (22) are rearranged the ensuing relation results:

$$(\mathrm{d}\varepsilon - \mathrm{d}\sigma/E_0)\{E_0 + \lambda(E_\infty \varepsilon - \sigma)\} = 0. \tag{ii}$$

However, because in this case inequality (19) holds it follows that $(\mathrm{d}\varepsilon - \mathrm{d}\sigma/E_0) \neq 0$ and hence if eqn (ii) is to hold then:

$$E_0 + \lambda(E_\infty \varepsilon - \sigma) = 0 \tag{iii}$$

or

$$\sigma = E_0/\lambda + E_\infty \varepsilon. \tag{iv}$$

Equation (iv) simply states that in this simple model the stress response obeys the kinematic hardening rule of Prager. This becomes more obvious upon substitution of the strain ε in terms of the plastic strain ε^p using eqn (v), i.e.

$$\varepsilon = \varepsilon^p + \sigma/E_0 \tag{v}$$

in which event eqn (iv) becomes:

$$\sigma = \sigma_Y + \beta\varepsilon^p \tag{vi}$$

where σ_Y is the yield stress and β the slope of the plastic stress strain curve! Specifically,

$$\sigma_Y = (E_0/\lambda)(1 - E_\infty/E_0) \tag{vii}$$

$$\beta = \lambda\sigma_Y(E_\infty/E_0). \tag{viii}$$

Without tedious repetition of the analysis, Case (b), i.e. inequality (20), gives rise to the complementary relation,

$$\sigma = -\sigma_Y + \beta\varepsilon^p \tag{ix}$$

which signifies the descending branch of the stress–plastic strain curve. See Fig. 1, where

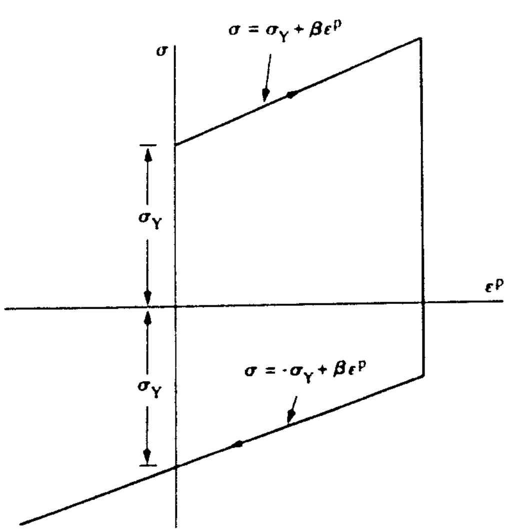

Fig. 1.

the stress–strain response, given by eqn (i) is illustrated. Evidently this is the case of classical linear kinematic hardening, derived from linear thermodynamics with the aid of intrinsic time, i.e. eqn (18). This is patently contrary to Fazio's claim that eqn (17) leads to an elastic response.

3. FAZIO'S ERRONEOUS ANALYSIS

Fazio's error lies in not recognizing the constraint of his inequality (19) and thereby realizing that eqn (iii) holds true. He proceeds to write eqn (i) in the form:

$$\mathrm{d}\sigma\{E_0+\lambda(E_\infty\varepsilon-\sigma)\} = E_0\{E_0+\lambda(E_\infty\varepsilon-\sigma)\}\,\mathrm{d}\varepsilon \tag{x}$$

and divides both sides of eqn (xi) by the term:

$$\{E_0+\lambda(E_\infty\varepsilon-\sigma)\}$$

not recognizing that this term is zero, to obtain:

$$\mathrm{d}\sigma = \frac{E_0+\lambda(E_\infty\varepsilon-\sigma)}{E_0+\lambda(E_\infty\varepsilon-\sigma)}\,\mathrm{d}\varepsilon \tag{24}$$

which is Fazio's eqn (24) with the plus sign option. Fazio then cancels terms to obtain:

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon \tag{28}$$

not realizing that he is dividing zero by zero. His eqn (24) reads:

$$\mathrm{d}\sigma = (0/0)E_0\,\mathrm{d}\varepsilon. \tag{xi}$$

What is more astonishing is that Fazio is oblivious to the fact that his derived eqn (28) *violates inequality* (19) *which was the basis of his derivation.*

This situation, the reader will agree, is somewhat unfortunate.

Remark 1. The physically opaque analysis (which can lead to error), which results from using the total strain as an independent variable, can be avoided if one works with the plastic strain instead as proposed by Valanis (1980). To this end use of eqn (v) in eqn (17) gives:

$$\sigma = \sigma_Y(\mathrm{d}\varepsilon^p/\mathrm{d}z)+\beta\varepsilon^p \tag{xii}$$

where σ_Y and β are given in eqns (vii) and (viii) respectively and:

$$\mathrm{d}z = |\mathrm{d}\varepsilon^p|. \tag{xiii}$$

For loading $\mathrm{d}\varepsilon^p > 0$, $\mathrm{d}\varepsilon^p/\mathrm{d}z = 1$, and eqn (vi) follows. For unloading and reverse loading $\mathrm{d}\varepsilon^p < 0$, $\mathrm{d}\varepsilon^p/\mathrm{d}z = -1$, and eqn (ix) follows. If $\mathrm{d}\varepsilon^p = 0$ then $\mathrm{d}\varepsilon^p/\mathrm{d}z$ is indeterminate and eqn (xiii) cannot give the stress. But now,

$$\mathrm{d}\sigma = E_0\,\mathrm{d}\varepsilon \tag{xiv}$$

so that the response is elastic as expected. So everything fits and is so simple. However, one must be willing to see the simplicity rather than search for fictitious complications that are not really there.

Remark 2. Equation (xiii) is a special case, the far more general constitutive relation:

$$\sigma = \sigma_Y(\mathrm{d}\varepsilon^p/\mathrm{d}z) + \int_0^z E(z-z')\,\mathrm{d}\varepsilon^p/\mathrm{d}z'\,\mathrm{d}z'. \tag{xv}$$

Equation (xii) is obtained from eqn (xv) by setting $E(z) = \text{constant}$. We note that eqn (xv) may be written in the form,

$$\sigma - \alpha = \sigma_Y(\mathrm{d}\varepsilon^p/\mathrm{d}z) \tag{xvi}$$

where α is the "back" stress and:

$$\alpha = \int_0^z E(z-z')\,\mathrm{d}\varepsilon^p/\mathrm{d}z'\,\mathrm{d}z'. \tag{xvii}$$

Equation (xvii) is, in fact, one of the main contributions of the endochronic theory to classical plasticity in that it is a demonstration that the back stress is a functional of the history of the plastic strain (Valanis, 1980).

DISCUSSION

Fazio has written an entire paper on the basis of results obtained by the mathematical operation of dividing zero by zero, violating in the process the very inequality that was the basis of his derivation. All his negative conclusions in regard to endochronic plasticity are blatantly wrong. Such papers are unfortunate. A full paper whose purpose will be to reiterate the salient features of endochronic plasticity will be forthcoming.

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K. C. VALANIS†
University of Portland
Portland
OR 97203
U.S.A.

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APPENDIX

Equation (17) may be derived in straightforward fashion from linear irreversible thermodynamics. In this particular case the free energy density ψ is given in terms of two elastic constants E_0 and E_r, ($E_0 > E_r$), and one internal variable q. Thus:

$$\psi = (1/2)E_r\varepsilon^2 + (1/2)(E_0 - E_r)(\varepsilon - q)^2. \tag{A1}$$

† President Endochronics, 8605 Northwest Lakecrest Court, Vancouver, WA 98665, U.S.A.

Application of the thermodynamic equations, i.e.

$$\sigma = \partial\psi/\partial\varepsilon \tag{A2}$$

$$\partial\psi/\partial q + \eta\, \mathrm{d}q/\mathrm{d}z \tag{A3}$$

gives eqn (17) where η is the resistance coefficient and λ is given by eqn (A4):

$$\lambda = (E_0 - E_\infty)/\eta. \tag{A4}$$

Equation (17) is the crux of endochronic plasticity, but beyond that it is an envolution equation of invariant form, applicable to a wide class of materials—from viscoelastic to fully plastic—a class whose members differ constitutively only in their intrinsic time scale.